UMass Boston Problem Solving Seminar Due Thu, Sep 17, 2015 Fall 2015 Problem Set # 1 2015-09-05 16:24

- (1) Show that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for all integers $n \ge 0$.
- (2) The Fibonacci numbers are defined by F(0) = F(1) = 1, and F(n) = F(n-1) + F(n-2) for $n \ge 2$. Determine the last digit of F(2015) (e.g. the last digit of 2015 is 5).
- (3) Let F(n) be the *n*th Fibonacci number. Compute

$$F(2014)F(2016) - F^2(2015).$$

- (4) What is the maximum number of regions you can obtain if you cut the plane using 2015 straight lines? What is the maximum number of regions you can obtain if you cut the space using 2015 planes?
- (5) In Mathland, candies come only in packages of 5, and 7. What is the maximum number of candies that can't be obtained as a combination of several full packages? What if the packages have 13 and 31 candies?
- (6) On a table there are 2015 weights, of masses 1 g, 2 g, ..., 2015 g. Can you divide the weights in three groups of equal mass?
- (7) Can 2015 be written as

$$2015 = \frac{p_1! p_2! \dots p_m!}{q_1! q_2! \dots q_n!} ,$$

where $p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_n$ are prime numbers, not necessarily distinct?

(8) For positive integers n, let the numbers c(n) be determined by the rules c(1) = 1, c(2n) = c(n), and $c(2n+1) = (-1)^n c(n)$. Compute

$$\sum_{n=1}^{2015} c(n)c(n+2)$$

(9) What is the last digit in the decimal representation of

$$N = 1^{2015} + 2^{2015} + 3^{2015} + \dots + 2014^{2015} + 2015^{2015} ?$$

(10) Find all positive real numbers $a_1, a_2, \ldots, a_{2015}$ such that

$$a_1^3 + a_2^3 + \dots + a_k^3 = (a_1 + a_2 + \dots + a_k)^2$$

for all $k = 1, 2, \dots, 2015$.

(11) Can 2015 be written as

$$2015 = \pm 1^2 \pm 2^2 \pm \dots \pm k^2$$

for some value k and some choice of \pm signs?

(12) Let a be a real number such that $a + a^{-1}$ is an integer. Show that $a^{2015} + a^{-2015}$ is also an integer.